Warmup - Calc 1 & 2

Calculus Bee
Tues 4/23, 5 PM
TMC 244

① $\int \frac{e^x}{e^x+1}\,dx$

② $\int_{-\pi}^{\pi} \left(x^3\cos(x) + \sin^7(x)e^{\cos^4(x)} - 7\right) dx$

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① $u = e^x + 1$
$du = e^x\,dx$ $\Rightarrow$ $\int \frac{du}{u} = \ln|u| + C$
$= \boxed{\ln(e^x+1) + C}$

② $f(-x) = -f(x)$
odd

$\int_{-\pi}^{\pi} \left(\underbrace{x^3\cos(x)}_{0} + \underbrace{\sin^7(x)e^{\cos^4(x)}}_{0} - 7\right) dx$

$= \int_{-\pi}^{\pi} -7\,dx = -7(2\pi) = \boxed{-14\pi}$.

3.32 b $\quad 0 \le x + 3y \le 4$

$-2 \le y \le \underbrace{x^2 + 6xy + 9y^2}_{(x+3y)^2 = u^2}$

Let $u = x + 3y$
$v = y$

$x + 4y = (x+3y) + y = u + v$

$$\int_{u=0}^{4}\int_{v=-2}^{u^2} (u+v)\, du \wedge dv$$

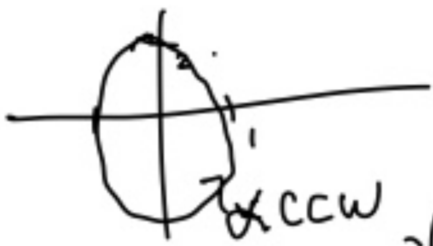
$du = dx + 3\,dy$
$dv = dy$

$du \wedge dv = (dx + 3dy) \wedge dy$
$= dx \wedge dy$

(3.21) $\oint_\alpha P\cdot ds$, $P=(2x^2, x-y)=(P_1,P_2)$

α ellipse $x^2+\frac{y^2}{4}=1$ oriented clockwise

$$\oint_\alpha P\cdot ds = -\int_{\substack{\alpha \\ CCW}} P\cdot ds = -\int_0^{2\pi} P(\alpha(t))\cdot\alpha'(t)\,dt$$



$\alpha(t)=(x(t),y(t))$

$=(\cos(t), 2\sin(t))$

$$= \int_0^{2\pi} P(x(t),y(t))\cdot(x'(t),y'(t))\,dt$$

...

Stokes'/Green's thm

$$\oint_{\alpha_{ccw}} P\cdot ds = \int_\alpha P_1\,dx + P_2\,dy$$

$$= \int_{\text{inside}} d(P_1\,dx + P_2\,dy) = \int_{\text{inside}} d(2x^2)\wedge dx + d(x-y)\wedge dy$$

$$= \int_{\text{inside}} 4x\,\underbrace{dx\wedge dx}_{0} + (dx - \underbrace{dy)\wedge dy}_{0}$$

$$= \iint_{\text{inside}} dx\wedge dy = \text{area(inside)}$$

$$= \underbrace{\pi ab}_{\text{area of ellipse}} = \pi(1)(2)$$

$$= \boxed{2\pi}.$$

original integral $= \boxed{-2\pi}$

Vector Field version

$$-\oint_{\alpha \text{ ccw}} P \cdot ds = -\iint (\text{curl } P) \cdot \underbrace{\hat{n}\, dA}_{\text{oriented area}}$$

unit normal of inside surface $\vec{k}$ -

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P_1 & P_2 & 0 \end{vmatrix} = 0 - 0 + k\left(\partial_x P_2 - \partial_y P_1\right)$$

$$= -\iint_{\text{inside}} \left[\underbrace{(P_2)_x}_{x-y} - \underbrace{(P_1)_y}_{2x^2} \right] dx\, dy$$

$$= -\iint_{\text{inside}} (1 - 0)\, dx\, dy = -\text{area (inside)}$$

$$= \boxed{-2\pi}.$$

Summary of Integral Calculations

① 1-dimensional integral (integrals over curves)

ⓐ Scalar functions integrated over a curve.

$$\int f\, ds \qquad f : \to \mathbb{R}$$

ⓘ $\int_a^b f(x)\, dx$ = integral of function over the interval $[a, b]$.

(ii) α is a general curve in $\mathbb{R}^n$

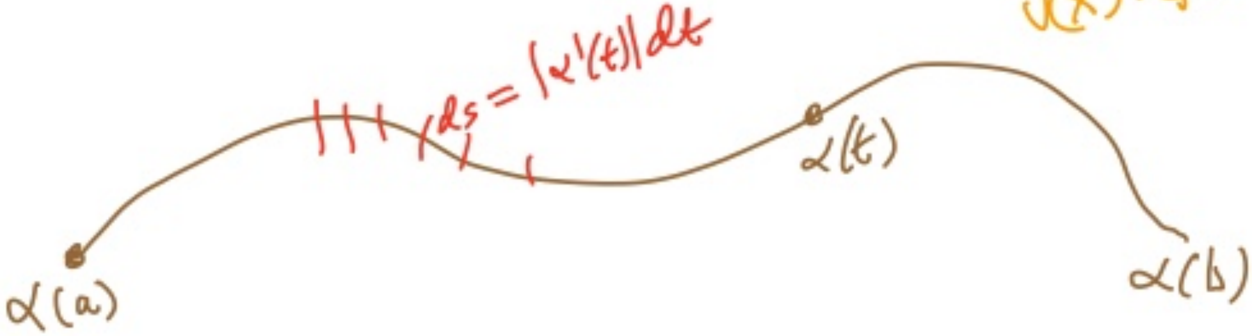
$\alpha(t) \quad a \le t \le b$

f is a scalar function on $\mathbb{R}^n$

$$\int_\alpha f\,ds = \int_a^b f(\alpha(t))\underbrace{|\alpha'(t)|\,dt}_{ds}$$

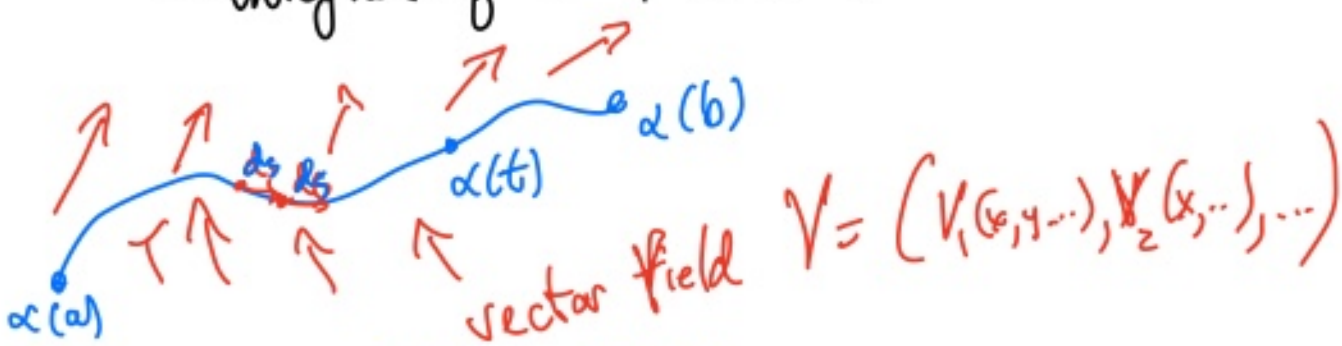
$$= \int_a^b f(x(t), y(t), \ldots)\ |\alpha'(t)|\,dt$$

$\sqrt{(x')^2+(y')^2+\ldots}$



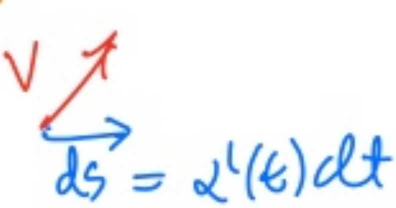
(b) Vector functions integrated over a curve (line integral)

= integral of a 1-form over a curve.



$$\oint_\alpha V \cdot \overbrace{ds}^{\text{vector } ds = \alpha'(t)dt}$$

dot product.



$V \cdot ds$ measures the component of V in the ds direction, multiplied by $|ds|$.

$$\hookrightarrow \oint_\alpha V(x(t),y(t),\dots)\cdot(x'(t),y'(t),\dots)\,dt$$

$$= \int_a^b \left(V_1(x(t),y(t)\dots)x'(t) + V_2(x(t),y(t)\dots)y'(t)+\dots\right) dt$$

Differential form version

$$= \int_\alpha (V_1\,dx + V_2\,dy + \dots)$$

Short cuts to help:

ⓐ (i) $\int_a^b F'(x)\,dx = F(b) - F(a)$ FTC

if you can find an antiderivative $F'(x) = f(x)$.

ⓑ $\oint_\alpha V\cdot ds = \int_\alpha V_1 dx + V_2 dy \dots$

(If we can find g such that $V = \nabla g$) $\Leftrightarrow$ (If we can find g such that $dg = V_1 dx + V_2 dy \dots$)

$= g(\text{end pt}) - g(\text{beg pt}).$ FTCLI

FTC for line integrals.

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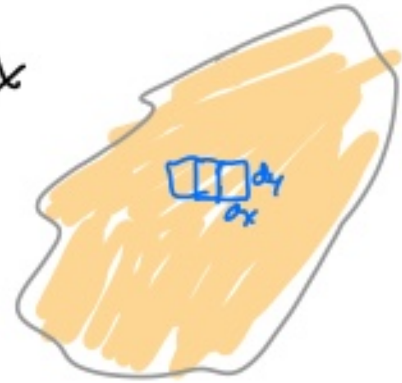
other shortcut — if curve is in a loop $\Rightarrow$ use Greens/Stokes Thm.

② Two-dimensional integrals:

ⓐ Integral of a scalar function over a surface.

ⓘ $\iint\limits_{\substack{\text{region} \\ \text{in } xy \text{ plane}}} f(x,y)\, dy\, dx$



ⓘⓘ integral over surface S in higher dimensions.

$$\int\limits_S f(x,y,z,\ldots)\, \underbrace{dA}_{\text{piece of surface area}}$$

"dS"

example of S surface with curve boundary.

To calculate, we will parametrize the surface

$$(x(u,v),\ y(u,v),\ z(u,v), \ldots)$$

$\_ \leq u \leq \_$

$\_ \leq v \leq \_$

we'll calculate

$$dA = \left| \begin{matrix} \text{cross product} \\ \text{of } u \text{ \& } v \text{ vectors} \end{matrix} \right| du\, dv$$

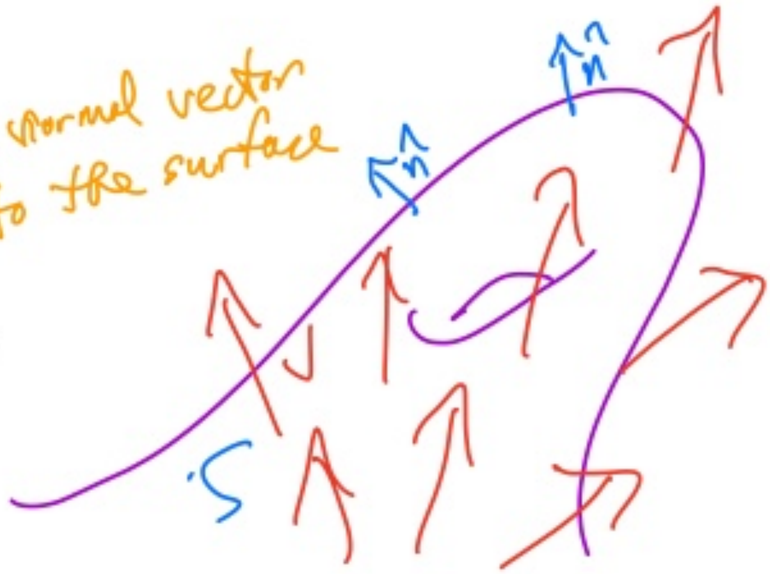
$$dA = \left| (x_u, y_u, z_u) \times (x_v, y_v, z_v) \right| du\, dv$$

(b) Flux of a vector field over a surface

$$\int_S V \cdot \underline{\hat{n}}\, dA$$

unit normal vector to the surface

measures the amount that the vector field flows out of the surface (in the $\hat{n}$ direction).



Vector fields & 2 forms.

$$V = \big(\underset{x\text{-comp}}{V_1(x,y,z)},\ \underset{y\text{-comp}}{V_2(x,y,z)},\ \underset{z\text{-comp}}{V_3(x,y,z)}\big)$$

$\iff$ 2 form

$$V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy$$

Think of $dy \wedge dz$ as a piece of area, times the vector perpendicular given by right hand rule.

$$\Rightarrow \int_S V\cdot\hat{n}\, dA = \int \underset{V_1(x,y,z)}{V_1}\, dy\wedge dz + V_2\, dz\wedge dx + V_3\, dx\wedge dy$$

surface

parametrize surface

$x(u,v)$
$y(u,v)$
$z(u,v)$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$
$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

Relation between surface integrals & line integrals.

$$\oint_{\alpha=\partial S} V\cdot ds \overset{\text{stokes}}{=} \int_S (\operatorname{curl} V)\cdot\hat{n}\, dA$$

closed curve

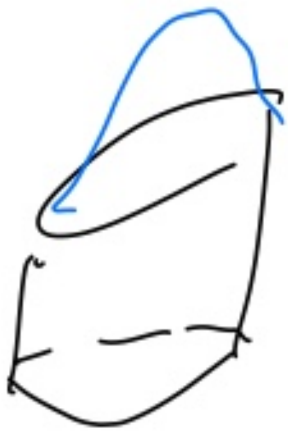
Be careful of orientation

$$= \int_{\alpha=\partial S} V_1\,dx + V_2\,dy + V_3\,dz = \int_S d(V_1\,dx + V_2\,dy + V_3\,dz)$$

③ 3-dimensional integrals:

Integral of scalar function over a region in $\mathbb{R}^3$.

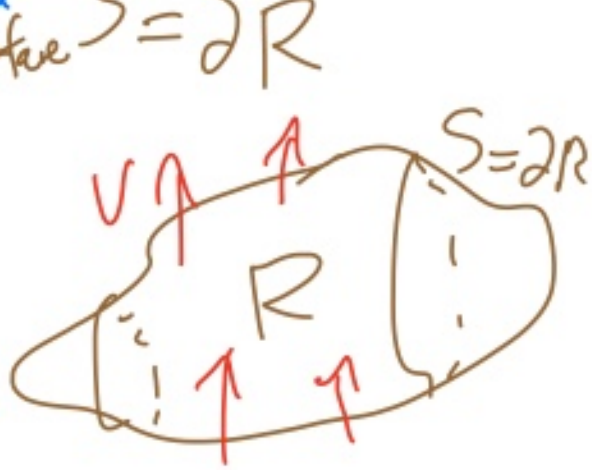
$$\iiint f(x,y,z)\, dz\, dy\, dx$$



Relation to surface integrals.

$$\int_{\substack{\text{closed surface} \\ S=\partial R}} V \cdot \hat{n}\, dA \overset{\text{divergence Thm.}}{=} \iiint_R (\operatorname{div} V)\, dz\, dy\, dx$$

$$\operatorname{div} V = \left(\partial_x V_1 + \partial_y V_2 + \partial_z V_3\right)$$



$$= \int_{\partial R} V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy$$

$$= \int_R d\Big(\qquad\qquad \Big)$$

$$= \int \big((V_1)_x + (V_2)_y + (V_3)_z \big)\, dx \wedge dy \wedge dz$$